Exercise Set Solutions #3

"Discrete Mathematics" (2025)

E1. **(a)** How many positive integers are there that divide 10^{40} or 20^{30} ?

> ***Solution:*** Note that $10^{40} = 2^{40} \cdot 5^{40}$ and $20^{30} = 2^{60} \cdot 5^{30}$. Consider the following sets
>
> $$A = \left\{a \in \mathbb{N} : a \mid 10^{40}\right\}, B = \left\{a \in \mathbb{N} : a \mid 20^{30}\right\}$$
>
> We have to count $|A \cup B|$. By inclusion-exclusion principle, we have
>
> $$|A \cup B| = |A| + |B| - |A \cap B|.$$
>
> Remember that the number of divisors of $n = p_1^{a_1} \cdots p_k^{a_k}$ is $(a_1 + 1) \cdots (a_k + 1)$. That means, $|A| = 41 \cdot 41 = 1681$ and $|B| = 61 \cdot 31 = 1891$. On the other hand,
>
> $$|A \cap B| = \left\{a \in \mathbb{N} : a \mid 10^{40} \text{ and } a \mid 20^{30}\right\} = \left\{a \in \mathbb{N} : a \mid 2^{40} \cdot 5^{30}\right\}$$
>
> That means, $|A \cap B| = 41 \cdot 31 = 1271$. Altogether, we obtain
>
> $$|A \cup B| = 1681 + 1891 - 1271 = 2301$$

(b) How many positive integers less than or equal to 385 are there such that they are not divisible by neither of the following numbers: $5, 7, 11$?

> ***Solution:*** $385 = 5 \cdot 7 \cdot 11$. Using Euler's function (recall its definition), we obtain that the result is
>
> $$385 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \cdot \left(1 - \frac{1}{11}\right) = 4 \cdot 6 \cdot 10 = 240$$

E2. Determine the number of permutations of the set $[n]$

(a) with exactly one fixed point, and

> ***Solution:*** Let A_i be the set of permutations which fix only i. Since A_i has no other fixed points apart from i, we obtain that its cardinality is equal to the cardinality of the set of all the permutations of an $(n-1)$-element set with no fixed point (in fact, the set of the other elements except i). But, from lecture notes (section 3.2), we already know that this number for $n - 1$ is
>
> $$(n-1)! \left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} \frac{1}{(n-1)!}\right).$$
>
> On the other hand, there are n choices for the fixed point and no two such permutations can coincide (since it is only one fixed point). That means, the desired result is
>
> $$n! \left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} \frac{1}{(n-1)!}\right).$$

(b) with exactly k fixed points.

Solution: Applying the same idea as for (1), we obtain

$$\binom{n}{k}(n-k)!\left(\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right).$$

E3. How many functions $f : [n] \to [n]$ are there that are nondecreasing? That is, they satisfy $i < j \Rightarrow f(i) \leq f(j)$.

Solution: Consider the following numbers.

$$\begin{aligned} x_1 &= f(1) - 1 \\ x_2 &= f(2) - f(1) \\ x_3 &= f(3) - f(2) \\ &\vdots \\ x_n &= f(n) - f(n-1) \\ x_{n+1} &= n - f(n). \end{aligned}$$

Then, we see that $\sum_{i=1}^{n+1} x_n = n - 1$ and each $x_i \geq 0$. Each such touple of $(x_1, \ldots, x_{n+1})$ gives us a nondecreasing function. Hence, the problem amounts to simply counting the set

$$\left\{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{\geq 0} \mid \sum_{i=1}^{n+1} x_i = n - 1\right\}.$$

The cardinality of this is equal to

$$\binom{(n+1)+(n-1)-1}{n-1} = \binom{2n-1}{n-1}$$

E4. Assume that $k > n$. Prove that the number of surjective functions from $[k]$ to $[n]$ is given by

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j(n-j)^k$$

Solution: Let A_j the set of functions from $[k]$ to $[n]$ taking values in a subset of cardinal j in $[n]$ (in other words, al the functions $f : [k] \to [n]$ such that $|f([k])| \leq j$). Thus, $A = \bigcup_j A_j$ represents all functions from $[k]$ to $[n]$. The number of surjective functions corresponds to $A \setminus \bigcup_{j=1}^{n-1} A_j$, therefore by inclusion-exclusion principle

$$|A\setminus\bigcup_{j=1}^{n-1} A_j| = |A|+\sum_{j=1}^{n-1}(-1)^{n-j}|A_j| = n^k+\sum_{j=1}^{n-1}(-1)^{n-j}\binom{n}{j}j^k = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^k = \sum_{j=0}^{n}\binom{n}{j}(-1)^j(n-j)^k$$

concluding.

E5. Prove the following.

(a) If $\varphi(n)$ divides $n-1$ then $n = p_1 \cdot \ldots \cdot p_r$ where $p_i \neq p_j$ for $i \neq j$.

Solution: We have that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is the prime factorization of n, then

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \left(\frac{n}{\prod_{i=1}^{k} p_i}\right) \prod_{i=1}^{k} (p_i - 1).$$

If - for example - $\alpha_i \geq 2$ then p_i^2 divides n, and thus p_i divides $\varphi(n)$. That implies that p_i divides n and $n-1$, which is a contradiction.

(b) $\varphi(n)$ is even for $n \geq 3$.

Solution: Observe from the formula. We have that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is the prime factorization of n, then

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \left(\frac{n}{\prod_{i=1}^{k} p_i}\right) \prod_{i=1}^{k} (p_i - 1).$$

Now whenever p_i is a prime number not equal to 2 then $p_i - 1$ will be even so n will be even. Hence, the only way $\varphi(n)$ is odd if $n = 2^k$ for some k (the other primes do not appear). But then $\varphi\left(2^k\right) = 2^k - 2^{k-1} = 2^{k-1}$. So $\varphi\left(2^k\right)$ is even only when $k = 1$ which is when $n = 2$.

(c) For every natural number n, we get

$$\sum_{d \mid n} \varphi(d) = n$$

where the sum is taken over all divisors d that divide n.

Solution: For any $d \mid n$, consider the set $[n]$ given by

$$S_d = \{r \in [n] \mid \gcd(r, n) = d\}.$$

Clearly, we have that

$$\bigsqcup_{d \mid n} S_d = [n].$$

Recall that the notation $d \mid n$ means that d "divides" n. This is simply the partitioning of n based on what the gcd of each number is with n. Note that the S_d are disjoint so that the sum

We will show that $|S_d| = \varphi(n/d)$. Consider the set

$$S'_d = \{r \in [n/d] \mid \gcd(r, n/d) = 1\}.$$

Note that $\gcd(r, n) = d \Leftrightarrow \gcd(r/d, n/d) = 1$. So we get a bijection from $f : S_d \to S'_d$ given by $r \mapsto r/d$. Clearly $|S'_d| = \varphi(n/d)$ and we prove the claim. This tells us that

$$\sum_{d \mid n} \varphi(n/d) = n.$$

But this proves the claim since the function $d \mapsto d/n$ gives us a bijection of the following set with itself.

$$\{d \in [n] | d|n\}$$

E6. **(a)** Suppose we have μ identical particles and n distinct energy levels, with $n \geq \mu$. In how many ways can we distribute the particles among the levels so that there is at most one particle per level?

Solution: For each distribution it is enough to determine which levels have one particle, as they can only have 0 or 1. This is equivalent to count how to choose a subset of μ levels from the n levels, which is $\binom{n}{\mu}$.

(b) Suppose we have μ distinct particles and n distinct energy levels, with $\mu \geq n$. In how many ways can we distribute the particles among the levels so that there is at least one particle per level?

Solution: If we list the particles from 1 to μ and the energy levels from 1 to n, the asked number corresponds to all surjective fuctions $f : [\mu] \to [n]$ which was obtain in problem $E4$.